Exercise 1

In each of the Exercises 1 through 3, use residues to find the inverse Laplace transform $f(t)$ corresponding to the given function $F(s)$. Do this in a formal way, without full justification,

$$F(s) = \frac{2s^3}{s^4 - 4}.$$

Ans. $f(t) = \cosh \sqrt{2}t + \cos \sqrt{2}t.$

Solution

Start by factoring the denominator.

$$F(s) = \frac{2s^3}{s^4 - 4} = \frac{2s^3}{(s^2+2)(s^2-2)} = \frac{2s^3}{(s+\sqrt{2}i)(s-\sqrt{2}i)(s+\sqrt{2})(s-\sqrt{2})}$$

Hence, there are four singularities.

$$s_1 = -\sqrt{2}i \quad s_2 = \sqrt{2}i \quad s_3 = -\sqrt{2} \quad s_4 = \sqrt{2}$$

The inverse Laplace transform is given by

$$f(t) = \sum_{n=1}^{4} \operatorname*{Res}_{s=s_n} [e^{st}F(s)].$$

We have

$$e^{st}F(s) = \frac{2s^3 e^{st}}{(s+\sqrt{2}i)(s-\sqrt{2}i)(s+\sqrt{2})(s-\sqrt{2})}.$$

Since all the factors in the denominator have multiplicity 1, s_1, s_2, s_3, and s_4 are simple poles, so the residues are of the form $\phi_n(s_n)$.

Let $\phi_1(s) = \dfrac{2s^3 e^{st}}{(s-\sqrt{2}i)(s+\sqrt{2})(s-\sqrt{2})}$. Then $\operatorname*{Res}_{s=s_1} [e^{st}F(s)] = \operatorname*{Res}_{s=s_1} \dfrac{\phi_1(s)}{s+\sqrt{2}i} = \phi_1(s_1) = \dfrac{1}{2}e^{-i\sqrt{2}t}$.

Let $\phi_2(s) = \dfrac{2s^3 e^{st}}{(s+\sqrt{2}i)(s+\sqrt{2})(s-\sqrt{2})}$. Then $\operatorname*{Res}_{s=s_2} [e^{st}F(s)] = \operatorname*{Res}_{s=s_2} \dfrac{\phi_2(s)}{s-\sqrt{2}i} = \phi_2(s_2) = \dfrac{1}{2}e^{i\sqrt{2}t}$.

Let $\phi_3(s) = \dfrac{2s^3 e^{st}}{(s+\sqrt{2}i)(s-\sqrt{2}i)(s-\sqrt{2})}$. Then $\operatorname*{Res}_{s=s_3} [e^{st}F(s)] = \operatorname*{Res}_{s=s_3} \dfrac{\phi_3(s)}{s+\sqrt{2}} = \phi_3(s_3) = \dfrac{1}{2}e^{-\sqrt{2}t}$.

Let $\phi_4(s) = \dfrac{2s^3 e^{st}}{(s+\sqrt{2}i)(s-\sqrt{2}i)(s+\sqrt{2})}$. Then $\operatorname*{Res}_{s=s_4} [e^{st}F(s)] = \operatorname*{Res}_{s=s_4} \dfrac{\phi_4(s)}{s-\sqrt{2}} = \phi_4(s_4) = \dfrac{1}{2}e^{\sqrt{2}t}$.

Summing the residues we obtain $f(t)$, the inverse Laplace transform of $F(s)$.

$$f(t) = \sum_{n=1}^{4} \operatorname*{Res}_{s=s_n} [e^{st}F(s)] = \frac{1}{2}e^{-i\sqrt{2}t} + \frac{1}{2}e^{i\sqrt{2}t} + \frac{1}{2}e^{-\sqrt{2}t} + \frac{1}{2}e^{\sqrt{2}t} = \frac{e^{\sqrt{2}t} + e^{-\sqrt{2}t}}{2} + \frac{e^{i\sqrt{2}t} + e^{-i\sqrt{2}t}}{2}$$

Therefore,

$$f(t) = \cosh \sqrt{2}t + \cos \sqrt{2}t.$$